

Reduction of the Statistical Power Per Event Due to Upper Lifetime Cuts in Lifetime Measurements

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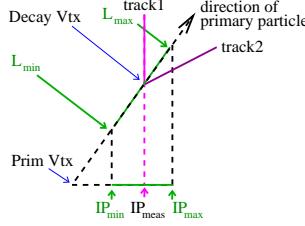
Abstract

A cut on the maximum lifetime in a lifetime fit does not only reduce the number of events, but it also, in some circumstances dramatically, decreases the statistical significance of each event. The upper impact parameter cut in the hadronic B trigger at CDF [1] [2] [3], which is due to technical limitations, has the same effect. In this note we describe and quantify the consequences of such a cut on lifetime measurements. We find that even moderate upper lifetime cuts, leaving event numbers nearly unchanged, can dramatically increase the statistical uncertainty of the fit result.

Keywords: *lifetime fit; lifetime cuts; impact parameter cuts; lifetime bias; hadronic B trigger; statistical power per event; CDF; B Physics*
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Figure 1: Given the 3-momenta of all particles in the decay, the cut on the Impact parameter of the decay products translates directly into a cut on the decaylength and hence on the lifetime of the primary particle. For clarity, the figure only illustrates the effect of an impact parameter cut on one of the decay products (the one going straight upwards).



1 Introduction

In this note we discuss the impact of an upper lifetime cut on the precision of a lifetime measurement. We find that even an upper lifetime cut that loses only a small fraction of the data can have dramatic consequences on the precision of the lifetime fit, due to a loss of statistical power of those events that pass the cut. The small loss of events due to a moderate upper lifetime cut is accompanied by a large loss of information, because not only a few events outside the allowed time window are lost, but also the information that there were only a few. This can have dramatic effects on the precision of the measurement. As shown below, an upper lifetime cut that loses 14 % of the data reduces the statistical significance per event by 72 %, so the combined effect on the statistical precision of the lifetime measurement is equivalent to losing 76 % of the data.

Such a cut on the maximum lifetime is for example implicitly applied in the hadronic trigger sample at CDF [1], where the trigger requires two tracks with a minimum impact parameter of $100 - 120 \mu\text{m}$ (depending on the exact configuration) and a maximum impact parameter of $1000 \mu\text{m}$ [2] [3]. These impact parameter cuts translate into upper and lower lifetime cuts, which differ event by event. This is illustrated in Figure 1.

2 Likelihood with lifetime cuts

We can write the probability to find an event with decay time t , given that it passed the trigger cuts, as:

$$P(t|t \in [t_{\min}, t_{\max}]) = \frac{\frac{1}{\tau} e^{-\frac{t}{\tau}}}{\int_{t_{\min}}^{t_{\max}} \frac{1}{\tau} e^{-\frac{t'}{\tau}} dt'} = \frac{\frac{1}{\tau} e^{-\frac{t}{\tau}}}{e^{\frac{-t_{\min}}{\tau}} - e^{\frac{-t_{\max}}{\tau}}} \quad (1)$$

where we ignore the effect of measurement errors. The total log-likelihood function for a set of N “ideal” two-body decays (no measurement uncertainties,

background, etc) is given by:

$$\begin{aligned}\log \mathcal{L} &= -N \log (\tau) \\ &\quad - \sum_{i=1}^N \left(\frac{t_i}{\tau} + \log \left(e^{-t_{\min i}/\tau} - e^{-t_{\max i}/\tau} \right) \right)\end{aligned}\tag{2}$$

where the index i labels the event, each of which has its measured decay time t_i and minimum and maximum lifetime cuts of $t_{\min i}$ and $t_{\max i}$.

Note that the only difference compared to the likelihood function without lifetime or impact parameter cuts is the term:

$$\log \mathcal{L}_{\text{ip}} = - \sum_{i=1}^N \log \left(e^{-t_{\min i}/\tau} - e^{-t_{\max i}/\tau} \right)\tag{3}$$

Including Gaussian event-by-event measurement errors, the PDF for an event with lifetime cuts t_{\min} and t_{\max} , and the measured lifetime t_0 , is given by:

$$\begin{aligned}P &= \frac{\int_0^{\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t_0)^2}{2\sigma_t^2}} dt}{\int_{t_{\min i}}^{t_{\max i}} \int_0^{\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t_0)^2}{2\sigma_t^2}} dt dt_0} \\ &= \frac{\frac{1}{\tau} e^{\frac{-t_0}{\tau} + \frac{1}{2} \frac{\sigma^2}{\tau^2}} F\left(\frac{t_0}{\sigma} - \frac{\sigma}{\tau}\right)}{\left[-e^{\frac{-t}{\tau} + \frac{1}{2} \frac{\sigma^2}{\tau^2}} F\left(\frac{t}{\sigma} - \frac{\sigma}{\tau}\right) + F\left(\frac{t}{\sigma}\right) \right]_{t=t_{\min i}}^{t=t_{\max i}}}\end{aligned}\tag{4}$$

with

$$F(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy\tag{5}$$

3 Calculating the Uncertainty of the Fit Result

The variance on the fit result can be estimated as the negative inverse of the 2nd derivative of the likelihood function, evaluated at the lifetime τ that maximises the likelihood:

$$\sigma^2 = V = -1 \left/ \frac{d^2 \log \mathcal{L}}{d\tau^2} \right|_{\tau=\tau_{\text{fit}}}\tag{6}$$

This formula assumes that near its minimum, the Likelihood curve can be approximated by a Gaussian. While this approximation is not perfect for lifetime fits, that have asymmetric errors, it is still very good as long as the event samples are sufficiently large, as shown in Section 5.

For simplicity, we ignore event-by-event measurement errors in our subsequent calculations. We will show in Section 5 that, for the purpose of estimating the

error on the fit result, this provides a very good approximation for the case of B lifetimes measured at CDF where the event-by-event measurement uncertainties are much smaller than the lifetime to be measured. The 1st derivative of the likelihood function defined in Equation 2 is:

$$\frac{d \log \mathcal{L}}{d\tau} = \frac{1}{\tau^2} \left(-N\tau + \sum_{i=1}^N (t_i - t_{\min i}) + \sum_{i=1}^N \frac{\Delta t_i}{e^{\Delta t_i/\tau} - 1} \right) \quad (7)$$

with $\Delta t_i \equiv t_{\max i} - t_{\min i}$

where we introduced $\Delta t_i = t_{\max i} - t_{\min i}$, the width of the time interval to which the i -th event is confined due to impact parameter, decay distance, or direct lifetime cuts. The 2nd derivative is:

$$\begin{aligned} \frac{d^2 \log \mathcal{L}}{d\tau^2} &= -\frac{2}{\tau} \frac{d \log \mathcal{L}}{d\tau} - \frac{1}{\tau^2} \left(N - \sum_{i=1}^N \left(\frac{\frac{1}{2}\Delta t_i/\tau}{\sinh(\frac{1}{2}\Delta t_i/\tau)} \right)^2 \right) \\ &= -\frac{2}{\tau} \frac{d \log \mathcal{L}}{d\tau} - \frac{N}{\tau^2} \left(1 - \left\langle \left(\frac{\frac{1}{2}\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle \right) \end{aligned} \quad (8)$$

where the angle bracket indicate taking the mean of the expression inside over all events. At the value of τ that maximises the likelihood, the first term of Equation 8 vanishes, and the variance is given by:

$$\sigma^2 = \frac{\tau^2}{N - \sum_{i=1}^N \left(\frac{\frac{1}{2}\Delta t_i/\tau}{\sinh(\frac{1}{2}\Delta t_i/\tau)} \right)^2} = \frac{\tau^2}{N} \cdot \frac{1}{1 - \left\langle \left(\frac{\frac{1}{2}\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle} \quad (9)$$

Note that the lower lifetime cut by itself does not have any impact on the statistical precision (apart from changing the number of events), it is the width of the time interval defined by the cuts, that matters; it is therefore the presence of an upper lifetime or impact parameter cut that affects the statistical precision per event.

4 Statistical Power Per Event

The right hand side of Equation 9 can be separated into two factors:

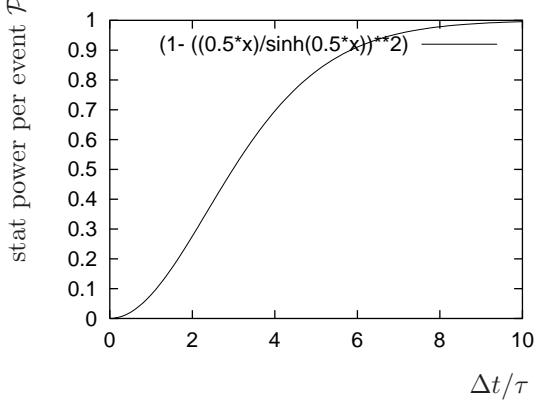
- The variance in the absence of any upper lifetime cut:

$$\frac{\tau^2}{N}. \quad (10)$$

- The correction factor due to the upper lifetime cut:

$$\frac{1}{1 - \left\langle \left(\frac{\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle} \quad (11)$$

Figure 2: Statistical Power per Event as a function of $\Delta t/\tau$.



\mathcal{P} is defined as

$$\mathcal{P} \equiv 1 - \left\langle \left(\frac{\frac{1}{2}\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle,$$

where Δt is the width of the time window defined by the lifetime cuts, τ is the lifetime to be measured.

So the change in statistical precision per event due to an upper lifetime cut is accounted for by making the following replacement for N :

$$N \longrightarrow N \cdot \left(1 - \left\langle \left(\frac{\frac{1}{2}\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle \right). \quad (12)$$

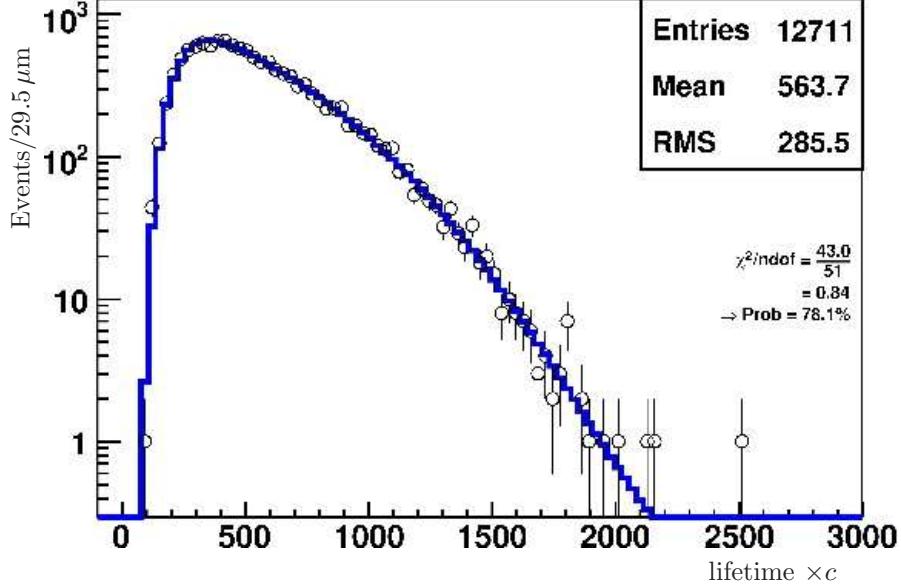
It makes therefore sense to define the *statistical power per event*, \mathcal{P} , as

$$\mathcal{P} \equiv 1 - \left\langle \left(\frac{\frac{1}{2}\Delta t/\tau}{\sinh(\frac{1}{2}\Delta t/\tau)} \right)^2 \right\rangle. \quad (13)$$

\mathcal{P} is 1 for events without upper lifetime cuts, and < 1 otherwise. It is defined such that N events with an upper lifetime cut (where N is the number of events after the cut has been applied) are statistically equivalent to $N \cdot \mathcal{P}$ events without an upper lifetime cut. Figure 2 shows the statistical power per event as a function of $\Delta t/\tau$, the time interval defined by the cuts divided by the lifetime. Upper lifetime cuts that seem harmless at first sight, because they have a small effect on the number of events, can have a dramatic impact on the statistical precision of the lifetime measurement due the reduction in statistical power per event. For example an upper lifetime cut leaving a time interval that is twice as wide as the mean lifetime to be measured ($\Delta t/\tau = 2$) retains $(1 - e^{-2}) = 86\%$ of the events, but the statistical power per event is reduced to 28%. The combined effect is equivalent to losing $100\% - 86\% \cdot 28\% = 76\%$ of the unbiased sample before the cut, rather than the naively expected 14%.

The CDF hadronic B trigger requires two tracks with impact parameters between $100 \mu\text{m}$ and $1000 \mu\text{m}$, which translate to different upper and lower lifetime cuts for each event, typically yielding a Δt between 1 and 3 times the B lifetime (this is an approximate number from studies of $B^0 \rightarrow \pi\pi$ decays). So each event in that sample is, for the purpose of lifetime measurements, only worth about 30% of an unbiased event. Note that this is true for lifetime measurements, only, and not for asymmetries or oscillation measurements, where it is the oscillation period that determines the scale Δt needs to be compared to, rather than the mean lifetime.

Figure 3: Fit to $\sim 13k$ MC-generated signal events. The shaping of the distribution due to the trigger manifests itself as a clear deviation from a straight line in this logarithmic plot. The fit describes the data well with a $\chi^2/\text{dof} = 0.8$. The result of $c\tau = 459.1^{+7.3}_{-7.1} \mu\text{m}$ is in good agreement with the input value of $462 \mu\text{m}$. The numerical error estimate of $^{+7.3}_{-7.1} \mu\text{m}$, calculated using the MINOS algorithm in MINUIT, agrees well with analytical value of $\pm 7.2 \mu\text{m}$ from Equation 9 derived [here](#).

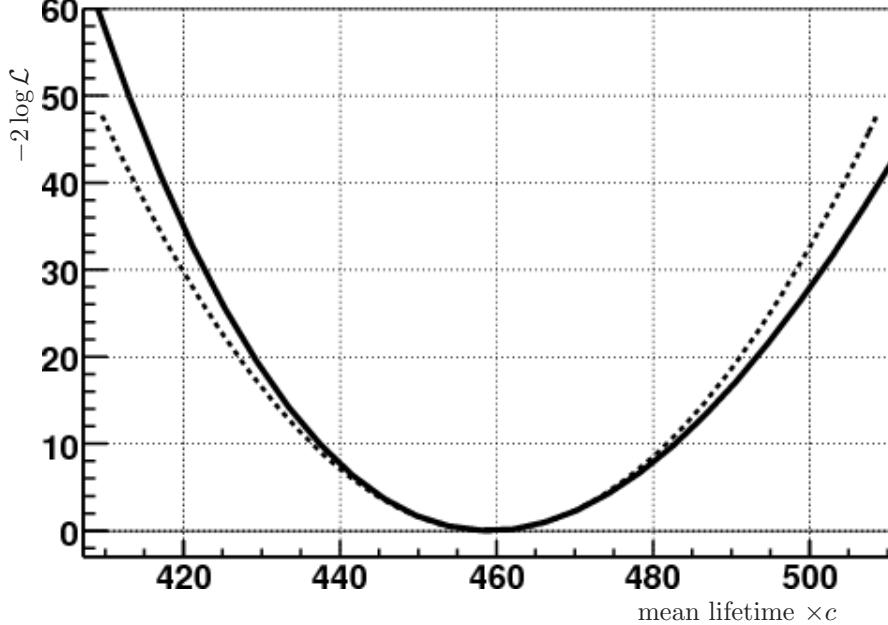


5 Monte Carlo Studies

In order to test how good the error estimate in Equation 9 is, given that this simplified formula ignores measurement errors and assumes the likelihood to be Gaussian near its minimum, it is compared to the error estimate from a MINUIT fit using a likelihood function that includes event by event errors. The errors are calculated using the MINOS algorithm within MINUIT. The fit is performed on simulated $B^0 \rightarrow \pi\pi$ events at CDF. The Monte Carlo simulation includes a detailed description of the CDF detector, including the hadronic B trigger, that requires two tracks with impact parameters between $100 \mu\text{m}$ and $1000 \mu\text{m}$. This requirement translates into upper and lower lifetime cuts, which differ event by event. The likelihood function used to fit the simulated data includes the trigger effects, and the event-by-event uncertainties in the lifetime measurement, as given in Equation 4. It is in the calculation of the acceptance limits where further detector effects are taken into account, in particular the difference between the fast-reconstructed online impact parameters used by the trigger, and the more precise offline reconstruction used in the actual lifetime measurement. The result of the fit to 12,711 signal events with an input lifetime of $c\tau = 462 \mu\text{m}$, is $c\tau = 459.1^{+7.3}_{-7.1} \mu\text{m}$. The agreement between the fit and the MC data is excellent, as can be seen in Figure 3.

The average statistical power per event, calculated using Equation 13, is 31 %.

Figure 4: The solid line is the likelihood curve for the fit to $\sim 13k$ MC-generated signal events. The broken line represents the parabolic approximation made implicitly in Equation 6.



Using this in Equation 9 yields an error estimate, *ignoring measurement errors*, of $7.2\mu\text{m}$, in good agreement with the numerical error estimate from the full likelihood. The solid line in Figure 4 represents the $-2 \log(\text{likelihood})$ curve for this fit, normalised to $-2 \log \mathcal{L} = 0$ at its minimum. The broken line represents the parabolic ($-2 \log(\text{Gaussian})$) approximation implicitly made in Equation 6. The agreement between the parabolic approximation and the actual likelihood for our data sample of $13k$ events is very good within the range relevant for calculating 1σ errors, i.e., for this normalisation, up to $(-2 \log \mathcal{L}) = 1$. The errors estimated using the full likelihood are $+7.3\mu\text{m}$, $-7.1\mu\text{m}$, in excellent agreement with the parabolic approximation of $\pm 7.2\mu\text{m}$. The difference between the two curves increases for larger $(-2 \log \mathcal{L})$ values, as the asymmetry in the lifetime errors becomes more pronounced, but it remains reasonable over the entire range of the plot, up to $(-2 \log \mathcal{L}) \approx 40$ and beyond. This implies that the parabolic approximation remains reasonable even for much smaller data samples containing only a few hundred events.

This gives us confidence that the formula derived in Equation 9 is correct, and that, for the purpose of estimating the statistical uncertainty of a lifetime measurement, the approximations we made are justified for B hadron lifetime measurements at CDF, where event samples are usually large and typical event-by-event lifetime errors are about 60 fs, small compared to B hadron lifetimes of about ~ 1.5 ps.

6 Summary

We quantified the statistical effect of upper lifetime cuts in lifetime measurements for the simplified case that the event-by-event lifetime errors are small compared to the lifetime to be measured.

We found that the effect of an upper lifetime cut is generally much more dramatic than the mere loss of events would suggest. The greatest impact of such a cut is a reduction in the statistical significance of each event for the purpose of lifetime measurements. For example an upper lifetime cut at twice the mean lifetime to be measured loses only 14% of events, but the statistical power per event is reduced by 72%. The combined effect is equivalent to a reduction in sample size by a factor 4, thus doubling the statistical error.

We verified our calculation using simulated $B^0 \rightarrow \pi\pi$ events at CDF. In the trigger scenario used as an example, the statistical power per event is reduced by 69% due to the upper impact parameter cuts applied by the trigger.

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